# Preferences and utility

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# Introduction

We talked in previous lectures about the difference between cooperative and non-cooperative game theory.

We now focus on non-cooperative game theory where the sets of actions of *individual* players are the primitives of the games.

We thus focus on strategic interactions between self-interested, independent agents:

- Chess: the game between two opponents
- Cold war: the game between U.S.A. and Soviet Union
- Rock-paper-scissors

# The focus on a single player

To rigorously analyze such strategic interactions between separate agents (e.g., individuals, firms, countries) we need to define

**Preferences**: what does each individual strive for in the interaction

If we can express these preferences through a real-valued function we gain analytical tractability:

**Utilities**: a real-valued function expressing a player's preferences

# Preferences

Let $x \in X$ be the set of decision alternatives for a player

**Definition: binary relation**

A *binary relation* $\succeq$ on a set $X$ is a non-empty subset $P \subset X \times X$. We write $x \succeq y$ if and only if $(x, y) \in P$.

$x \succeq y$: "the player weakly prefers $x$ over $y$"

Define two associated binary relations:

$x \succ y$: "the player strictly prefers $x$ over $y$"

$x \sim y$: "the player is indifferent between $x$ and $y$"

# Modern assumptions on preferences

1. Completeness
2. Transitivity
3. Continuity
4. Independence of irrelevant alternatives

For each of these assumptions we will see

- an intuitive explanation
- a definition
- a critical discussion

# Completeness

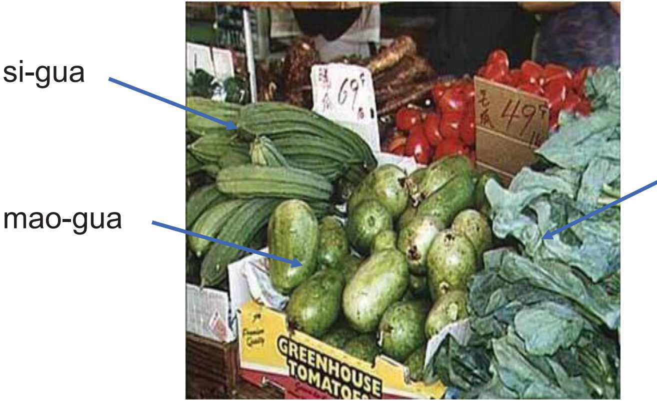
If a consumer is choosing between two bundles $x$ and $y$ one of the following possibilities hold:

- $x \succ y$ – she prefers $x$ to $y$
- $y \succ x$ – she prefers $y$ to $x$
- $x \sim y$ – she is indifferent between $x$ and $y$

**Axiom 1: Completeness**

$\forall x, y \in X : x \succeq y$ or $y \succeq x$ or both

# Completeness: Choices over Chinese vegetables




# Completeness: Observations

Consumers / agents / humans often find it impossible to rank some options...
or do so in a way that is in some sense wrong for them.

Decision making takes time and effort, we are often

- uninformed
- uncertain
- unable to evaluate what a product is and does
- subject to biases / inattention (Kahneman: Thinking fast and slow)

The initial framework is not based on psychological research.
Behavioral economics is, we will come to that at a later stage.

# Transitivity

If a consumer is choosing between three bundles $x, y, z$ with $x \succ y$ and $y \succ z$, then:

- $x \succ z$

**Axioms 2: Transitivity**

$\forall x, y, z \in X$ : if $x \succeq y$ and $y \succeq z$, then $x \succeq z$

# Transitivity: Choices over cars

 $\succ$  (because it is faster)

 $\succ$  (because it carries many people)

 $\succ$  (because it is easier to park)

# Transitivity: Choices over cars

# Transitivity: Observations

Consumers / agents / humans often find it difficult to rank choices coherently.

Ranking several options often depends on

- many dimensions (speed, space, CO2 emission, ...)
- different needs

# Continuity

If a consumer is choosing between three bundles $x, y, z$ with $x \succ y$ and $y$ is very similar to $z$, then:

- $x \succ z$

Let $\succeq$ be a rational preference ordering on $X$. For $x \in X$ define the subsets of alternatives that are (weakly) *worse/better* than $x$ :

$$W(x) = \{y \in X : x \succeq y\}$$
$$B(x) = \{y \in X : y \succeq x\}$$

**Axiom 3: Continuity**

$\forall x \in X : B(x)$ and $W(x)$ are closed sets.

# Continuity: Choices over breakfast

To understand the axiom of continuity it is easier to think of the rate of consumption of a good:

Suppose:

- 100g Müsli/ day $\succ$ 200g bananas per day

Then:

- 100g Müsli / day $\succ$ 201g bananas per day

# Utility function

It is very convenient for the mathematical modeling of an agent with binary relation $\succeq$ if we can find a real-valued function whose expected value the agent aims to maximize.

**Definition**

A **utility function** for a binary relation $\succeq$ on a set $X$ is a function $u : X \to \mathbb{R}$ such that

$$u(x) \geq u(y) \iff x \succeq y$$

**Proposition 1a**

There exists a utility function for each complete, transitive, positively measurable, and continuous preference ordering on any closed set.

# Utility function – for countable sets

It is very convenient for the mathematical modeling of an agent with binary relation $\succeq$ if we can find a real-valued function whose expected value the agent aims to maximize.

**Definition**

A **utility function** for a binary relation $\succeq$ on a set $X$ is a function $u : X \to \mathbb{R}$ such that

$$u(x) \geq u(y) \iff x \succeq y$$

**Proposition 1b**

There exists a utility function for every transitive and complete preference ordering on any countable set.

*Proof.* Exercise, for those of you inclined.

# Let's play a game!

A fair coin is tossed until head shows for the first time:

- If head turns up first at $1^{st}$ toss you win 1 CHF
- If head turns up first at $2^{nd}$ toss you win 2 CHF
- If head turns up first at $3^{rd}$ toss you win 4 CHF
- ...
- If head turns up first at $k^{th}$ toss you win $2^{k-1}$ CHF

*You have a ticket for this lottery. For which price would you sell it?*

# Utility $\neq$ Payoff

If you only care about expected gain:

$$\begin{aligned}\mathbb{E}[\text{lottery}] &= \frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 + \frac{1}{8}\cdot 4 + ... \\ &= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + ... \\ &= \infty\end{aligned}$$

- Bernoulli suggested in 1738 the theory of diminishing marginal utility of wealth.
- Further, the need for utility characterization under uncertainty arose.

This laid the foundation for ***expected utility theory***.

# Expected-utility theory

Let $T = \{\tau_1, ...\tau_m\}$ be a finite set and let $X$ consist of all probability distributions on $T$:

$$X = \Delta(T) = \{x = (x_1, ...., x_m) \in \mathbb{R}^m_+ : \sum_{k=1}^{m} x_k = 1\}$$

That is $X$ is the unit simplex in $\mathbb{R}^m$.

***Can we define a utility function in this setting?***

# Independence of irrelevant alternatives

If a consumer is choosing between two bundles $x$ and $y$ with $x \succ y$, then for any $z$:

- $x + z \succ y + z$

**Axiom 4: Independence of irrelevant alternatives**

$\forall x, y, z \in X, \forall \lambda \in (0, 1)$:

$$x \succ y \quad \Rightarrow \quad (1-\lambda)x + \lambda z \succ (1-\lambda)y + \lambda z$$

# Independence of irrelevant alternatives: Example

To understand the axiom of independence of irrelevant alternatives it is again easier to think of the rate of consumption of a good:

Suppose that an agent prefers Asian food over Italian but also values an occasional Italian dish. Then:

$$\text{sushi} \succ \text{pizza} \not\Rightarrow (1-\lambda)\cdot\text{sushi}+\lambda\cdot\text{wontons} \succ (1-\lambda)\cdot\text{pizza}+\lambda\cdot\text{wontons}$$

# Independence of irrelevant alternatives: Observations

"The problem with *independence of irrelevant alternatives* is that it is reductionist." (Don Saari)

That means, it assumes that any decision can be broken down into its smallest parts!

# Bernoulli function / von Neumann-Morgenstern utility function

If $\succeq$ is a binary relation on $X$ representing the agent's preferences over lotteries over $T$. If there is a function $v : T \to \mathbb{R}$ such that

$$x \succeq y \iff \sum_{k=1}^{m} x_k v(\tau_k) \geq \sum_{k=1}^{m} y_k v(\tau_k)$$

then

$$u(x) = \sum_{k=1}^{m} x_k v(\tau_k)$$

defines a utility function for $\succeq$ on $X$.

The assumption that preferences can be expressed in this form is called the *expected utility hypothesis*. $v$ is called a *Bernoulli function*.

# Existence of von Neumann-Morgenstern utility function

1. Axiom 1: Completeness
2. Axiom 2: Transitivity
3. Axiom 3: Continuity
4. Axiom 4: Independence of irrelevant alternatives

**Theorem (von Neumann-Morgenstern)**

Let $\succeq$ be a complete, transitive and continuous preference relation on $X = \Delta(T)$, for any finite set $T$.
Then $\succeq$ admits a utility function $u$ of the expected-utility form if and only if $\succeq$ meets the axiom of independence of irrelevant alternatives.

# Allais paradox

The set of prices in CHF is $T = \{0; 1,000,000; 5,000,000\}$.

- Which probability do you prefer:
  $x_1 = (0.00; 1.00; 0.00)$ or $x_2 = (0.01; 0.089; 0.10)$?

- Which probability do you prefer:
  $x_3 = (0.90; 0.00; 0.10)$ or $x_4 = (0.89; 0.11; 0.00)$?

Most people report: $x_1 \succ x_2$ and $x_3 \succ x_4$.

# Allais paradox

The set of prices in CHF is $T = \{0; 1,000,000; 5,000,000\}$.

- $x_1 = (0.00; 1.00; 0.00)$ or $x_2 = (0.01; 0.089; 0.10)$
- $x_3 = (0.90; 0.00; 0.10)$ or $x_4 = (0.89; 0.11; 0.00)$

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Suppose $(v_0, v_{1M}, v_{5M})$ is a Bernoulli function for $\succeq$.

Then $x_1 \succ x_2$ implies:

$$v_{1M} > .01 \cdot v_0 + .89 \cdot v_{1M} + .1 \cdot v_{5M}$$

$$.11 \cdot v_{1M} - .01 \cdot v_0 > .1 \cdot v_{5M}$$

now add $.9 \cdot v_0$ to both sides:

$$.11 \cdot v_{1M} + .89 \cdot v_0 > .1 \cdot v_{5M} + .9 \cdot v_0$$

But this implies $x_4 \succ x_3$, a contradiction!

# Sure thing principle

**Sure thing principle (Savage)**

A decision maker who would take a certain action if he knew that event $B$ happens and also if he knew that $not - B$ happens, should also take the same action if he know nothing about $B$.

**Lemma**

Assume that everything the decision maker knows is true then *sure thing principle is equivalent to independence of irrelevant alternatives*.

# Sure thing principle: Savage (1954)

*"A businessman contemplates buying a certain piece of property. He considers the outcome of the next presidential election relevant. So, to clarify the matter to himself, he asks whether he would buy if he knew that the Democratic candidate were going to win, and decides that he would. Similarly, he considers whether he would buy if he knew that the Republican candidate were going to win, and again finds that he would. Seeing that he would buy in either event, he decides that he should buy, even though he does not know which event obtains, or will obtain, as we would ordinarily say. It is all too seldom that a decision can be arrived at on the basis of this principle, but except possibly for the assumption of simple ordering, I know of no other extralogical principle governing decisions that finds such ready acceptance."*

# Translation invariance

Given a Bernoulli function $v$ for given preferences $\succeq$ let:

$$v' = \alpha + \beta v$$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$.
Then $v'$ is also a Bernoulli function for another utility function

$$u' = \alpha + \beta u$$

*Expected utility functions are unique up to a positive affine transformation.*

# Ordinal, cardinal utility functions, and "utils"

**Ordinal utility function.** A utility function where differences between $u(x)$ and $u(y)$ are meaningless. Only the fact that, for example, $u(x) \geq u(y)$ is meaningful. An ordinal utility function can be subjected to any increasing transformation $f(u)$ which will represent the same preferences $\succeq$.

**Cardinal utility function.** A utility function where differences between $u(x)$ and $u(y)$ are meaningful as they reflect the *intensity* of preferences. Cardinal utility functions are only invariant to positive affine transformations.

**"Utils".** An even stronger statement would be that there is a fundamental measure of utility, say one "util". Such a utility function is not invariant to any transformation.

# Comparing utility: within person

Can the following statement be represented by ............... utility function?

1. "She likes x less than z"
2. "She likes x over z twice as much as y over z"
3. "She likes x five times more than y"

| | 1. | 2. | 3. |
|---|---|---|---|
| **Ordinal utility function** | yes | no | no |
| **Cardinal utility function** | yes | yes | no |
| **"Utils"** | yes | yes | yes |

# Comparing utility: interpersonal

**Interpersonal comparability (IC).** A utility function where utility differences between players make "sense".

| | |
|---|---|
| **Ordinal utility function** | possibly IC |
| **Cardinal utility function** | possibly IC |
| **"Utils"** | $\Rightarrow$ IC |

Suppose we have cardinal utility functions that are IC for agent 1 and 2, $u_1, u_2$. Transform them by some non-affine increasing transformation $f$ resulting in $v_1 = f(u_1), v_2 = f(u_2)$.
Then $v_1, v_2$ are no longer cardinal but are IC.

*Note:* Utility functions that are ordinal and do not allow for interpersonal comparisons do not contain more information than preference relations.

# Comparing utility: interpersonal

Can the following statement be represented by ................ utility function?

1. “Warren Buffet values 1000 CHF less than a starving child values 1000 CHF”
2. “Eve would pay 10 CHF (utils) for the chocolate, Sarah would pay 5 CHF (utils)”
3. “Mother loves d1 more than d2. Father loves d2 more than d1”

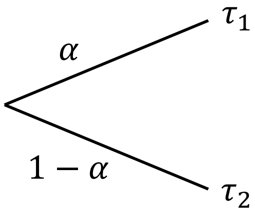
| | 1. | 2. | 3. |
|---|---|---|---|
| **Ordinal utility function** | | | |
| **Cardinal utility function** | | | |
| **“Utils”** | | | |

**Comparing utilities between agents is almost always impossible.**

**Comparing utilities between agents implies some welfare statement / judgment.**

# Utility and risk

Define a lottery:




The lottery is a fair gamble if and only if $\alpha \cdot v(\tau_1) = (1-\alpha) \cdot v(\tau_2)$.

# Risk neutrality

**Definition: risk neutral**

An agent is risk-neutral if and only if he is indifferent between accepting and rejecting all fair gambles, that is for all $\alpha, \tau_1, \tau_2$:

$$\begin{aligned} \mathbb{E}[u(\text{lottery})] &= \alpha \cdot v(\tau_1) + (1-\alpha) \cdot v(\tau_2) \\ &= u(\alpha \cdot \tau_1 + (1-\alpha) \cdot \tau_2) \end{aligned}$$

An agent is risk-neutral if and only if he has a linear von Neumann-Morgenstern utility function.

# Risk aversion

**Definition: risk averse**

An agent is risk averse if and only if he rejects all fair gambles, that is for all $\alpha, \tau_1, \tau_2$:

$$\begin{aligned} \mathbb{E}[u(\text{lottery})] &= \alpha \cdot v(\tau_1) + (1-\alpha) \cdot v(\tau_2) \\ &< u(\alpha \cdot \tau_1 + (1-\alpha) \cdot \tau_2) \end{aligned}$$

Recall that a function $g(\cdot)$ is strictly concave if and only if

$$g(\lambda\alpha + (1-\lambda)\beta) > \lambda g(\alpha) + (1-\lambda)g(\beta)$$

An agent is risk averse if and only if he has a strictly concave utility function.

# Risk seekingness

**Definition: risk seeking**

An agent is risk seeking if and only if he strictly prefers all fair gambles, that is for all $\alpha, \tau_1, \tau_2$:

$$\begin{aligned} \mathbb{E}[u(\text{lottery})] &= \alpha \cdot v(\tau_1) + (1-\alpha) \cdot v(\tau_2) \\ &> u(\alpha \cdot \tau_1 + (1-\alpha) \cdot \tau_2) \end{aligned}$$

An agent is risk seeking if and only if he has a strictly convex utility function.

# Some final remarks

- If you believe that people have preferences, under “reasonable” axioms we can translate them into utility functions.
    - Nevertheless, we should always be aware that our analysis is based on several assumptions / axioms.

- Money is not equal to utility (recall diminishing marginal utility).

- Preferences do not have to be self-regarding (“homo oeconomicus”).

THANKS EVERYBODY

Keep checking the website for new materials as we progress:
http://gametheory.online/project_show/9