# COOPERATIVE GAME THEORY

Heinrich H. Nax

hnax@ethz.ch

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Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

# Game theory

Non-cooperative game theory

- No binding contracts can be written
- Players are individuals
- Main solution concepts:
  - Nash equilibrium
  - Strong equilibrium

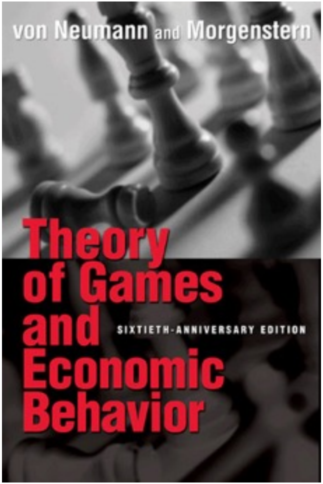
Cooperative game theory

- Binding contract can be written
- Players are individuals and coalitions of individuals
- Main solution concepts:
  - Core
  - Shapley value

# A noncooperative game (normal-form)

- players: $N = 1, 2, ..., n$ (finite)
- actions / strategies: (each player chooses $s_i$ from his own finite strategy set; $S_i$ for each $i \in N$)
  - resulting strategy combination: $s = (s_1, ..., s_n) \in (S_i)_{i \in N}$
- payoffs: $u_i = u_i(s)$
  - payoff outcome of the game

# Cooperative games and GAME theory



# Cooperative Game: MODEL ingredients

- players: $N = 1, 2, ..., n$ (finite)
- coalitions: $C \subseteq N$ form
  - resulting in a coalition structure $\rho$

NEED TO SPECIFY ...coalition formation and allocation of payoffs $\phi$...
THEN RESULT

- payoffs: $u_i = u_i(\rho, \phi)$
  - payoff outcome of the game

# Characteristic function form (A MODEL by von NEUMANN-MORGENSTERN 1944)

- CFG defined by 2-tuple $G(v, N)$
- players: $N = 1, 2, ..., n$ (finite, fixed population)
- coalitions: disjoint $C \subseteq N$ form resulting in a coalition structure/ partition $\rho$
  - $\emptyset$ is an empty coalition
  - $N$ is the grand coalition
  - The set of all coalitions is $2^N$
  - $\rho$ is the set of all partitions
- $v$ is the characteristic function form that assigns a worth $v(C)$ to each coalition and $v(\emptyset) = 0$
  - v: $2^N \rightarrow R$

# 3-player example

- $v(i)=0$
- $v(1,2)=v(1,3)=0.5$
- $v(2,3)=0$
- $v(N)=1$

# Transferable-utility cooperative game

- CFG defined by 2-tuple $G(v, N)$
- Outcome: partition $\rho = \{C_1, C_2, ..., C_k\}$ and payoff allocation/imputation $\phi = \{\phi_1, ..., \phi_n\}$
- in each C, $\sum_{i \in C} \phi_i \leq v(C)$ – *feasibility*

# 3-player example

- Outcome 1: {(1,2),3} and {(0.25,0.25),0}
- Outcome 2: {N} and {0.25,0.25,0.5}
- Outcome 3: {N} and {0.8,0.1,0.1}

# Superadditivity assumption

If two coalitions C and S are disjoint ($S \cap C = \emptyset$), then
$v(C) + v(S) \leq v(C \cup S)$
i.e. "mergers of coalitions weakly improve the worth of the coalitions"
This implies that, for all $\rho \in P, v(N) \geq \sum_{C \in \rho} v(C)$
i.e. "efficiency of the grand coalition"

# 3-player example

$$v(N) > v(1,2) = v(1,3) > v(2,3) = v(1) = v(2) = v(3)$$

# The core (Gillies 1959)

- The core X of G(v,n) consists of all outcomes where the grand coalition forms and payoff allocations are such that
- $\sum_{i \in N} \phi_i = v(N)$ – Pareto-efficient
- And, for all $C \subset N$,
- $\sum_{i \in C} \phi_i \geq v(C)$ – unblockable
  - individual rational: $\phi_i \geq v(i)$ for all $i$
  - coalitional rational: $\sum_{i \in C} \phi_i \geq v(C)$ for all $C$

# 3-player example

- Outcome 1: {(1,2),3} and {(0.25,0.25),0}
- Outcome 2: {N} and {0.25,0.25,0.5}
- Outcome 3: {N} and {0.8,0.1,0.1}

# Properties of the core

- A system of weak linear inequalities define the core, which is therefore closed and convex.
- The core can be empty, non-empty, or large.

# Core empty

- $v(i) = 0$
- $v(i,j) = 0.9$
- $v(N) = 1$

# Core unique

- $v(i) = 0$
- $v(i,j) = 2/3$
- $v(N) = 1$

# Core large

- $v(i) = v(i,j) = 0$
- $v(N) = 1$

# ndareva-shapley theorem

- The core of a game is nonempty if and only if the game is "balanced" (Bondareva 1963,Shapley 1967)
- Balancedness:
  Balancing weight: Let $\alpha(C) \in [0, 1]$ be the balancing weight attached to any $C \in 2^N$
  Balanced family: A set of balancing weights $\alpha$ is a balanced family if
  - for every $i$, $\sum_{C \in 2^N : i \in C} \alpha(C) = 1$

  Balancedness then requires that, for all balanced families,
  - $v(N) \geq \sum_{C \in 2^N} \alpha(C) v(C)$

# Limitations of The core

# 1. Core empty

- $v(i) = 0$
- $v(i,j) = 5/6$
- $v(N) = 1$

# 2. Core non-empty but very inequitable (1, 0, 0)

- $v(i) = v(2,3) = 0$
- $v(N) = v(1,2) = v(1,3) = 1$

# 3. Core large (any split of 1)

- $v(i) = v(i, j) = 0$
- $v(N) = 1$

# Shapley value: a Normative solution concept

- Given some N, then for any v an acceptable allocation/value x*(v) should satisfy
- Efficiency. $\sum_{i \in N} x *_i (v) = v(N)$
- Symmetry. if, for any two players $i$ and $j$, $v(S \cup i) = v(S \cup j)$ for all $S$ not including $i$ and $j$, then $x *_i (v) = x *_j (v)$
- Dummy player. if, for any $i$, $v(S \cup i) = v(S)$ for all $S$ not including $i$, then $x *_i (v) = 0$
- Additivity. If u and v are two characteristic functions, then $x * (v + u) = x * (v) + x * (u)$

# Shapley value

The function

$$\phi_i(v) = \sum_{S \in N, i \in S} \frac{(|S|-1)!(n-|S|)!}{n!}[v(S) - v(S \setminus \{i\})]$$

is the unique function satisfying all four axioms for the set of all games.

# Alternative axioms

**Young (1985) proved that a set of equivalent, more attractive axioms is**

- Efficiency. $\sum_{i \in N} x*_i(v) = v(N)$
- Symmetry. if, for any two players $i$ and $j$, $v(S \cup i) = v(S \cup j)$ for all $S$ not including $i$ and $j$, then $x*_i(v) = x*_j(v)$
- Monotonicity. If u and v are two characteristic functions and, for all $S$ including $i$, $u(S) \geq v(S)$, then $x*_i(u) \geq x*(v)$

# Interpretation

The Shapley Value is a player's average marginal contribution:

- For any $S$: $i \in S$,
- $MC(S) = v(S) - v(S \setminus i)$

# 1. Core empty

- $v(i) = 0$
- $v(i,j) = 5/6$
- $v(N) = 1$

**Shapley value**

$(1/3, 1/3, 1/3)$

# 2. Core non-empty but very inequitable (1, 0, 0)

- $v(i) = v(2,3) = 0$
- $v(N) = v(1,2) = v(1,3) = 1$

**Shapley value**

$(4/6, 1/6, 1/6)$

# 3. Core large (any split of 1)

- $v(i) = v(i, j) = 0$
- $v(N) = 1$

**Shapley value**

$(1/3, 1/3, 1/3)$

# Room-entering story

## Roth (1983)

suppose the players enter a room in some order and that all $n!$ orderings of the players in $N$ are equally likely. Then $\phi_i(v)$ is the expected marginal contribution made by player $i$ as she enters the room. To see this, consider any coalition $S$ containing $i$ and observe that the probability that player $i$ enters the room to find precisely the players in $S-i$ already there is $(s-1)!(n-s)!/n!$. Out of $n!$ permutations of $N$ there are $(s-1)!$ different orders in which the first $s-1$ players can precede $i$, and $(n-s)!$ different orders in which the remaining $n-s$ players can follow, for a total of $(s-i)!(n-s)!$ permutations in which precisely the players $S-i$ precede $i$.

# Relationship Between core and shapley value

Put simply, none...

- when the core is non-empty, the SV may lie inside the core or outside the core
- when the core is empty, the SV is still uniquely determined

# Other cooperative models

# nonTransferable-utility cooperative game

- CFG defined by 2-tuple $G(v, N)$
- Outcome: partition $\rho = \{C_1, C_2, ..., C_k\}$ implies a payoff allocation/imputation such that $\phi_i = f_i(C_i)$

> "Agents have preferences over coalitions". There are no side-payments and the worth of the coalition cannot be distributed.

# Matching markets

# Stable Marriage/Matching problem

A 2-sided market with $n$ men on one side, and $n$ women on the other.

- Each man $m_i$ has individual preferences (e.g. $w_1 > w_2 > ... > w_n$) over the women
- Each woman $w_i$ has individual preferences (e.g. $m_n > m_1 > ... > w_{n-1}$) over the men

We want to establish a stable matching of couples (man-woman) such that there exists no alternative couple where both partners prefer to be matched with each other rather than with their current partners.

# Deferred acceptance (Gale-shapley Theorem 1962)

For any marriage problem, one can make all matchings stable using deferred acceptance. Use in practice (e.g. Roth & Sotomayor 1990, Roth et al. ...):

- Resource allocations for hospitals
- Organ transplantations
- School admissions
- Assigning users to servers in distributed Internet services
- ...

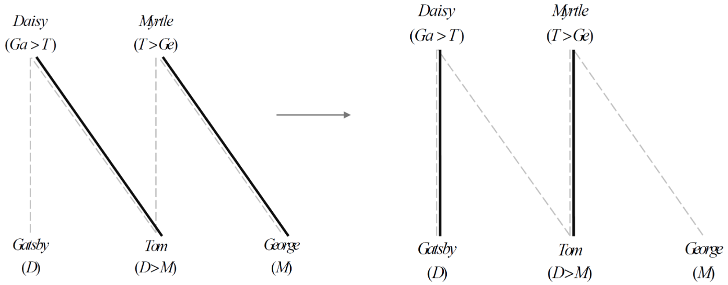
# deferred acceptance in pseudo-code

- Initialize: all $m_i \in M$ and all $w_i \in W$ are single.
- Engage: Each single man $m$ "proposes" to his preferred woman $w$ to whom he has not yet proposed.
  - If $w$ is single, she will become "engaged" with her preferred proposer.
  - Else $w$ is already engaged with $m'$
    - If $w$ prefers her preferred proposer $m$ over $m'$, then $(m, w)$ become engaged and $m'$ becomes single
    - Else $(m', w)$ remain engaged.
  - All proposers who do not become engaged remain single.
- If a single man exists, repeat Engage; Else move to terminate
- Terminate: "Marry" all engagements.

# Proof sketch

- Women "trade up" until everyone is engaged, which is when they all get married.
- No singles can remain, because every man would eventually propose to every woman as long as he remains single, and once proposed to, a single woman becomes engaged.
- The resulting matching is stable!
- Proof: Suppose the algorithm terminates so that there exists a pair $(m, w)$ whose partners are engaged to $w'$ and $m'$ respectively, but not to each other. It is not possible for both $m$ and $w$ to prefer each other over their engaged partner, because
    - If $m$ prefers $w$ over $w'$, then he proposed to $w$ before he proposed to $w'$. At that time,
        - If $w$ got engaged with $m$, but did not marry him, then $w$ must have traded up and left $m$ for someone she prefers, and therefore cannot prefer $m$ over $m'$.
        - Else, if $w$ did not get engaged with $m$, then she was already with someone she prefers to $m$, and can therefore not prefer $m$ over $m'$.
    - Hence, either $m$ prefers $w'$ over $w$, or $w$ prefers $m'$ over $m$.

# What went wrong in the great gatsby



# The assignment game

See you next week
THANKS EVERYBODY