BARGAINING

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Feb 20, 2019: Part 2 -- Lecture 3

Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Plan for today

1. Bargaining applications
2. Cooperative bargaining solution
3. Noncooperative bargaining program
4. Experimental bargaining

Lecture logic

Topic

- Introduce bargaining
- Illustrations/ applications
- Bridge cooperative and noncooperative game theory (again...)

Appeal

- Bargaining is ubiquitous
- May be useful in real life
- Illustrates the idea of the “Nash program”

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Examples of bargaining

Markets:

- Individuals (buyer/seller)
- Strategies (bid/ask certain prices)
- Outcome (profits/losses)

jcrs.com

Splitting:

- Players (partners)
- Strategies (demands)
- Outcome (a split)

mirror.co.uk

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Bargaining in real-world markets

Bombay Stock Exchange

Stock market:

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L'Inde Fantome
(L. Malle 1969)

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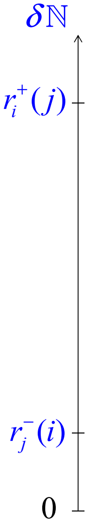
Bargaining over what?

Buyers/sellers and their willingness to pay/accept

Buyer $\boldsymbol{i} \in \boldsymbol{B}$ and seller $\boldsymbol{j} \in \boldsymbol{S}$ look for partners ($|\boldsymbol{B}| = |\boldsymbol{S}| = \boldsymbol{N}$) – each seller owns exactly one good and each buyer wants exactly one good

- Buyer $\boldsymbol{i}$ is willing to pay at most $\boldsymbol{r_i^+(j)} \in \boldsymbol{\delta}\mathbb{N}$ for the product of seller $\boldsymbol{j}$
- Seller $\boldsymbol{j}$ is willing to accept at least $\boldsymbol{r_j^-(i)} \in \boldsymbol{\delta}\mathbb{N}$ to sell his product to buyer $\boldsymbol{i}$

where $\boldsymbol{\delta} > \mathbf{0}$ is the minimum unit ('dollars')



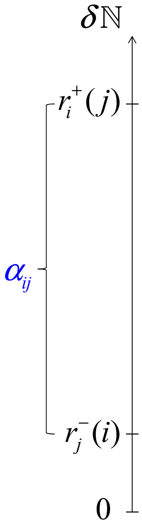
Bargaining over the match value

The match value for the pair (i,j) is

$$\alpha_{ij} = (r_i^+(j) - r_j^-(i))_+$$

Let $\alpha = (\alpha_{ij})_{i\in F, j\in W}$

Normalization.
Let's normalize this value to the 'unit-pie' $\alpha_{ij} = 1$ for some (i,j).



Is there any economic activity more basic than two people dividing a pie?

The pie could symbolize the gains from trade in a market, the surplus generated within a firm, or the benefit from writing a joint paper on economics. Supposing that the nature of the split does not affect the pie's total size, this is a case in which distribution and efficiency is thought not to conflict. Surely, sensible people will come to some agreement rather than backing away from the transaction empty-handed. This argument has permeated economic thinking at least since Edgeworth [1881], and is sometimes referred to as neoclassical bargaining theory (see, e.g., Harsanyi [1987]).

from T. Ellingsen (1997): The Evolution of Bargaining Behavior.

The basic bargaining model

- *Ingredients:*
  - Multiple parties/players
  - A common gain/pie
  - No central authority
  - Bargaining ensues
  - Some outcome is reached

From the analyst's point of view, how do we model this as a "game"?

Two approaches

- *Cooperative:*
  - Multiple parties/players
  - Coalitions form/contract is written
  - Normative axioms are established
  - Outcome is identified
  - Outcome is implemented
- *Noncooperative:*
  - Multiple parties/players
  - Bargaining follows some rules
  - Players act strategically
  - Bargaining takes place
  - Outcome is implemented

Examples

- *Cooperative:*
  - Twins share presents
  - They have identical preferences
  - Twins agree on a splitting rule
  - Sharing fifty-fifty is the only fair rule accepted by both
  - Presents are divided in equal halves
  - Outcome is implemented

- *Noncooperative:*
  - A buyer and a seller meet on the market
  - They have different preferences
  - Buyers make offers
  - Sellers make counteroffers
  - Both try to get the most out of the deal
  - If an offer is accepted, they deal
  - If not, no deal

Compare with our 'cooperative solutions' (Lecture 2)

- *Shapley value:*
  - All players could agree on the axioms
  - They could write an agreement that the SV is implemented
  - Then the outcome would be implemented

- *Core:*
  - When the SV lies inside the core, this seems stable
  - However, as the SV may lie outside the core
  - Or when the core is empty
  - Then there would exist coalitions that perhaps would break the deal

The first formal model (Nash again!)

2-person cooperative bargaining

Nash (1953): Two-Person Cooperative Games. Econometrica 21.

Aside: there were earlier versions due to Edgeworth 1881, Zeuthen 1930 and von Neumann and Morgenstern 1944.

2-person cooperative bargaining

Two person sharing the unit-pie

Basic ingredients:

- players $N = \{1, 2\}$
- outside options $v(i) = o_i \in [0, 1)$ for both $i \in N$
- agreement value $v(N) = 1$

The aim:

- The goal is to reach an agreement (s_1, s_2) such that
- $s_1 + s_2 = 1$ – Pareto efficient
- $s_i \geq o_i$ for all i – Individually rational

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Nash bargaining 1

Individual preferences and ***normative postulates***:

- Agents have different preferences $u_i(c)$ s.t.
- $\partial u_i(c)/\partial c > 0$ and
- $\partial^2(u_i(c))/\partial c^2 < 0$
- The outcome that is reached should be "fair"!
- But what is fair?

If everything (including preferences and outside options) is identical, ... easy...
$50 : 50$ is fair.

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Nash bargaining 2

In general, there may be conflict between what is "fair" and what will be reached by strategic bargaining.

Nash program

Derive a framework for noncooperative bargaining, at the end of which the outcome is a *Nash equilibrium* (i.e. such that everyone's choice is optimal given the choices of others), and that outcome implements a *cooperative solution* concept.

Illustrating the Nash program

- Bargaining sets obtained from a bimatrix game
- Bargaining axioms
- The Nash bargaining solution
- Geometric characterization of the Nash bargaining solution

- Splitting a unit pie, concave utility functions
- The ultimatum game
- Alternating offers over several rounds
- Stationary strategies
- The Nash bargaining solution via alternating offers

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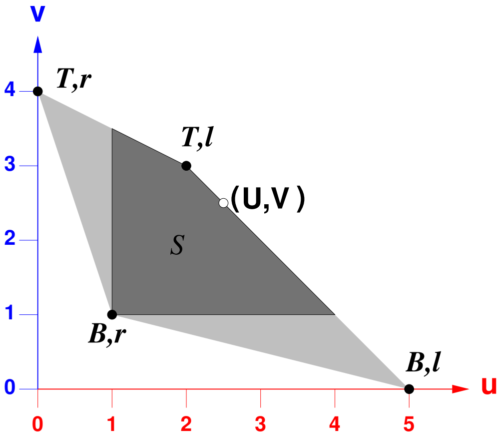
Toward a cooperative bargaining solution: “The Nash Bargaining Solution”

JF Nash (1950). ‘The Bargaining Problem’. ***Econometrica*** $\mathbf{18(2) : 155 - 162}$.

Bargaining set from a bimatrix game

I \ II	l	r
T	2, 3	0, $\boxed{4}$
B	$\boxed{5}$, 0	$\boxed{1}$, $\boxed{1}$



Axioms for **Bargaining Set** $\boldsymbol{S} \subset \mathbb{R}^{\mathbf{2}}$

- Threat point $(\boldsymbol{u}_{\mathbf{0}}, \boldsymbol{v}_{\mathbf{0}}) \in \boldsymbol{S}$,
  for all $(\boldsymbol{u}, \boldsymbol{v}) \in \boldsymbol{S}$: $\;\boldsymbol{u} \geq \boldsymbol{u}_{\mathbf{0}}, \boldsymbol{v} \geq \boldsymbol{v}_{\mathbf{0}}$.
- $\boldsymbol{S}$ is compact (bounded and closed)
- $\boldsymbol{S}$ is convex (via agreed joint lotteries)

Axioms for **Nash Bargaining Solution** $N(S)$

- $N(S) = (U, V) \in S$.
- **Pareto-optimality:** for all $(u, v) \in S$:

  $u \geq U$ and $v \geq V \;\Rightarrow\; (u, v) = (U, V)$
- **Invariance of utility scaling:** $a, c > 0$,

  $S' = \{(au + b, cv + d) \mid (u, v) \in S\} \;\Rightarrow\; N(S') = (aU + b, cV + d)$.
- **Symmetry:** if S is symmetric, then so is $N(S)$:

  If $(u, v) \in S$ implies $(v, u) \in S$, then $U = V$.
- **Irrelevance of unused alternatives:** If S, T are bargaining sets with the same threat point and $S \subset T$, then $N(T) \notin S$ or $N(T) = N(S)$.

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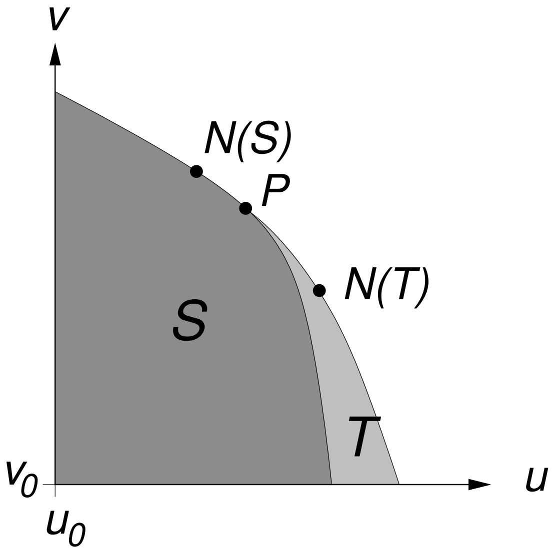
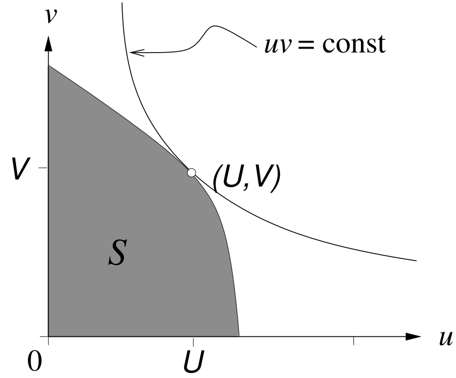
Irrelevance of unused alternatives



The Nash bargaining solution [Nash 1950]

Under the Nash bargaining axioms, every bargaining set S containing a point (u, v) with $u > u_0$ and $v > v_0$ has a unique solution $N(S) = (U, V)$.

(U, V) maximises the following product–
Nash product: $(U - u_0)(V - v_0)$ for $(U, V) \in S$.



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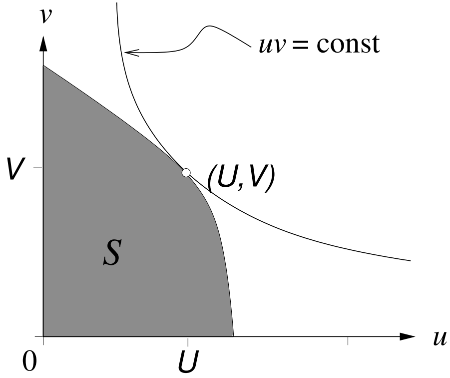

Nash bargaining solution - proof

- **Shift threat point** (u_0, v_0) to $(0, 0)$:
  replace S with $S' = \{(u - u_0, v - v_0) \mid (u, v) \in S\}$

$\Rightarrow$ Nash product maximised as UV (rather than $(U - u_0)(V - v_0)$)

- **re-scale utilities** so that $(U, V) = (1, 1)$:
  replace S with $S' = \{(u/U, v/V) \mid (u, v) \in S\}$.

- **consider** $T = \{(u, v) \mid u \geq 0,\ v \geq 0,\ u + v \leq 2\}$
  $N(T) = (1, 1)$, because T is a symmetric set, and $(1, 1)$ is the only symmetric point on the Pareto-frontier of T.

- Claim: $S \subseteq T \Rightarrow$ (by independence of irrelevant alternatives) $N(S) = N(T)$ because $(1, 1) \in S$.

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Proof that $S \subseteq T$

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Suppose exists $(\overline{u}, \overline{v}) \in S$, $(\overline{u}, \overline{v}) \notin T \;\Rightarrow\; \overline{u} + \overline{v} > 2$.

Idea: even if Nash product $\overline{u}\,\overline{v} \leq 1 = UV$, still $\boxed{uv > 1}$ for $(u, v) = (1 - \varepsilon)(1, 1) + \varepsilon(\overline{u}, \overline{v})$, contracting maximality of UV, where $(u, v) \in S$ by convexity of S.

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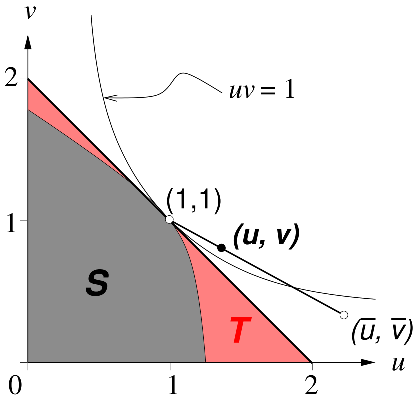

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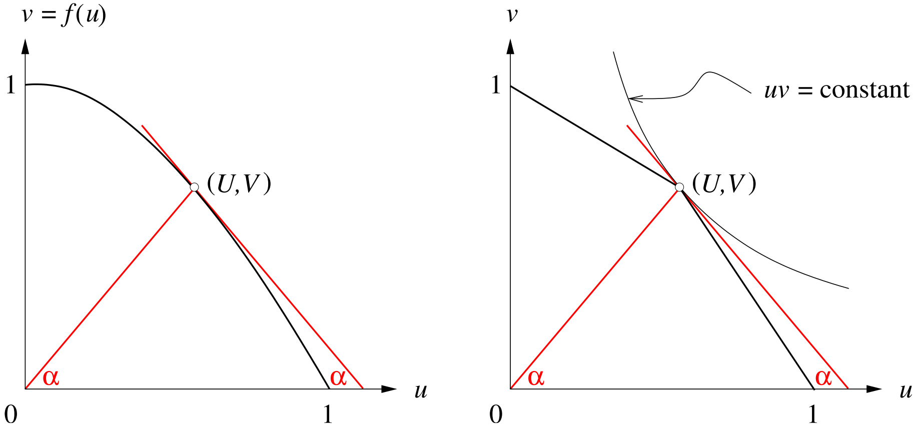
where $(u, v) \in S$ by convexity of S.

Indeed $uv > 1$ for sufficiently small ε because

$$
\begin{aligned}
uv &= (1 - \varepsilon + \varepsilon\bar{u})(1 - \varepsilon + \varepsilon\bar{v}) \\
&= (1 + \varepsilon(\bar{u} - 1))(1 + \varepsilon(\bar{v} - 1)) \\
&= 1 + \varepsilon(\boxed{\bar{u} + \bar{v} - 2} + \varepsilon(\bar{u} - 1)(\bar{v} - 1)) \\
&> 1 \quad \text{for sufficiently small } \varepsilon > 0 \text{ because } \boxed{\bar{u} + \bar{v} - 2 > 0}
\end{aligned}
$$

Geometric characterization of $\boldsymbol{U}, \boldsymbol{V}$



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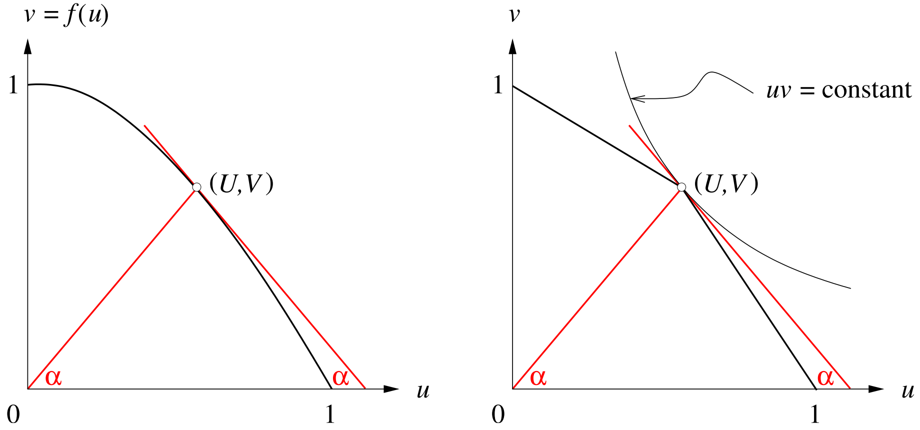
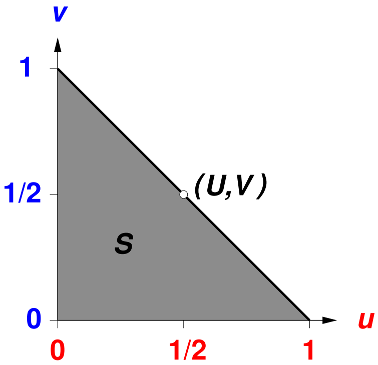


Splitting a unit pie

Suppose player I and player II have to split an amount (a "pie") of one unit into $\boldsymbol{x}$ for player I and $\boldsymbol{y}$ for player II, where

$$\boldsymbol{x} \geq \mathbf{0}, \quad \boldsymbol{y} \geq \mathbf{0}, \quad \boldsymbol{x} + \boldsymbol{y} \leq \mathbf{1}\,.$$

Then this defines in a simple way a bargaining set $\boldsymbol{S}$ if $\boldsymbol{u} = \boldsymbol{x}$ and $\boldsymbol{v} = \boldsymbol{y}$.



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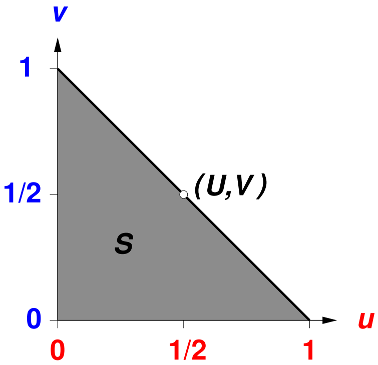
$$x \geq 0, \quad y \geq 0, \quad x + y \leq 1 .$$

Then this defines in a simple way a bargaining set S if $u = x$ and $v = y$.



Split pie with utility functions

More generally, assume the pie is split into x and y so that

player I receives $u(x)$, player II receives $v(y)$,

where $x \geq 0$, $y \geq 0$, $x + y \leq 1$. Here

player I has utility function $u : [0, 1] \to [0, 1]$

player II has utility function $v : [0, 1] \to [0, 1]$

with $u(0) = 0$, $u(1) = 1$, $v(0) = 0$, $v(1) = 1$,

and u and v increasing, continuous. and **concave**.

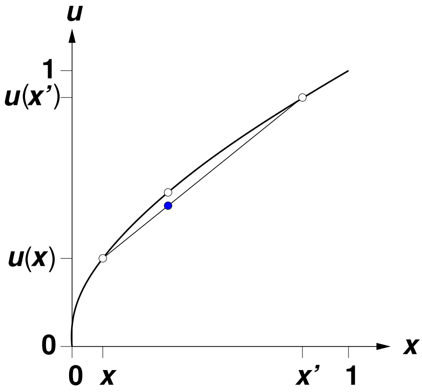
Concave utility functions

A concave utility function u has "diminishing returns". If u is differentiable this means $u'' \leq 0$, in general

$$(1-p)u(x) + pu(x') \leq u((1-p)x + px')$$

for all x, x' and $p \in [0, 1]$.

Example
$u(x) = \sqrt{x}$



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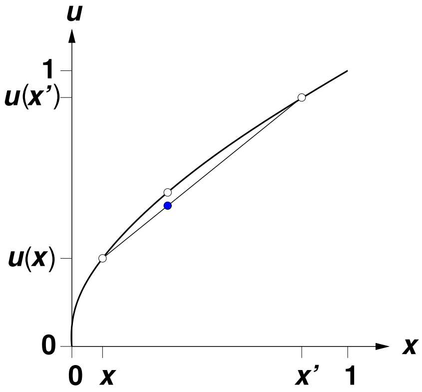


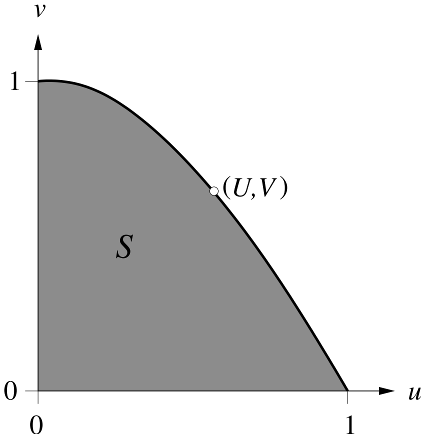
Example
$u(x) = \sqrt{x}$

Convex bargaining set

With concave u and v, the bargaining set S is convex,

$$S = \{(u(x), v(y)) \mid x \geq 0,\ y \geq 0,\ x + y \leq 1\}$$

Example
$u(x) = \sqrt{x}$
$v(y) = y$



Nash bargaining solution

Example $u(x) = \sqrt{x}, \quad v(y) = y$

Pareto-frontier = $\{(u(x), v(1-x) \mid 0 \le x \le 1\}$

The Nash bargaining solution maximizes

$$u(x)v(1-x) = \sqrt{x}(1-x) = x^{1/2} - x^{3/2}\,.$$

Derivative set to zero:

$$0 = \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{2}x^{-1/2}(1-3x),$$

that is, $x = 1/3$ = share for player I, and player II gets $y = 2/3$.

Utilities $(U, V) = (\sqrt{1/3}, 2/3) \approx (0.577, 0.667)$.

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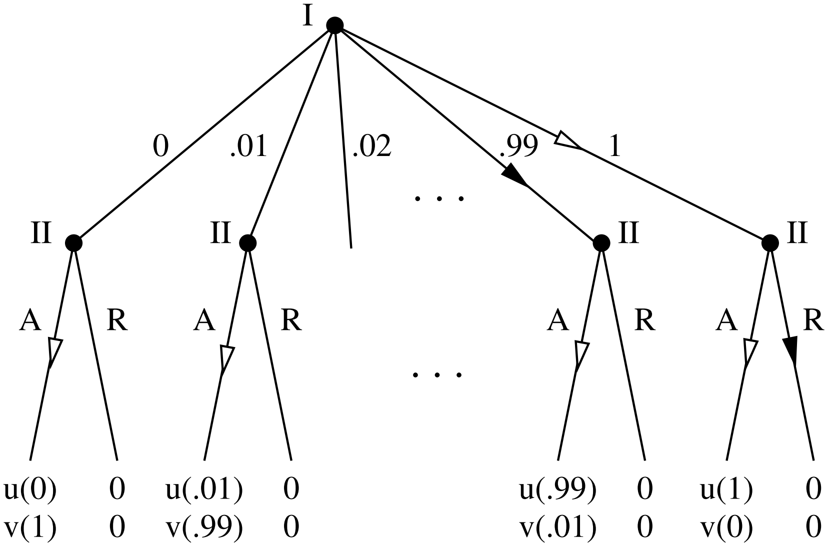
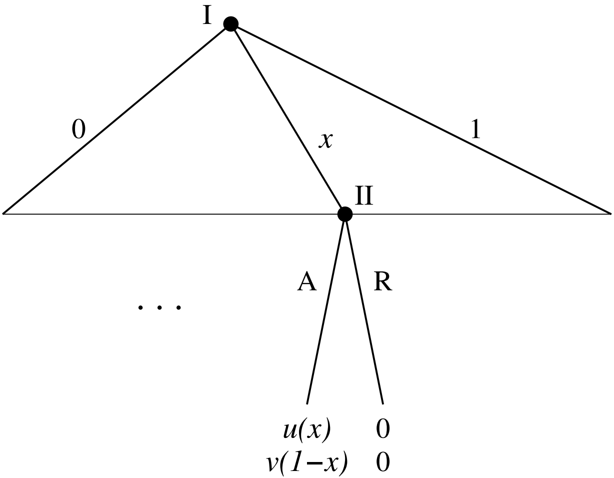
Toward noncooperative foundations: “The Rubinstein Bargaining Model”

A Rubinstein (1982). ‘Perfect Equilibrium in a Bargaining Model’. ***Econometrica*** $\mathbf{50(1) : 97 - 109}$.

The ultimatum game



Continuous version of the ultimatum game



SPNE: player I makes player II **indifferent** between accepting and rejecting, here with $x = 1$, but player II nevertheless accepts.

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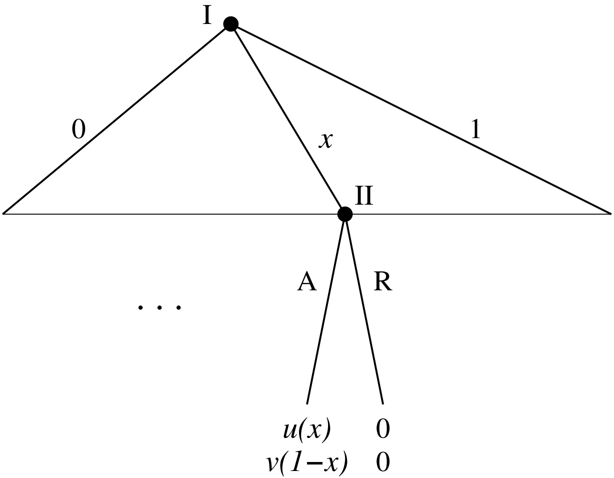

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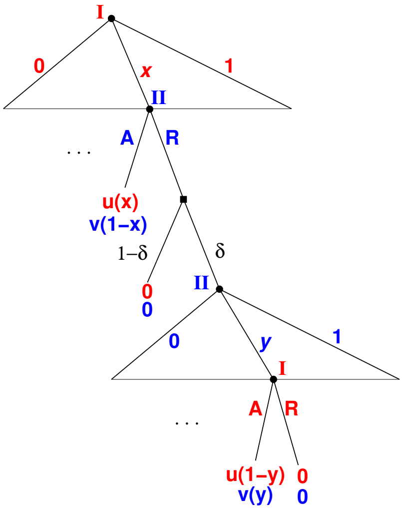
Bargaining in two rounds

$\delta =$ probability that negotiations continue,

$\mathbf{0} < \boldsymbol{\delta} < \mathbf{1}$.

Taking expected payoffs :



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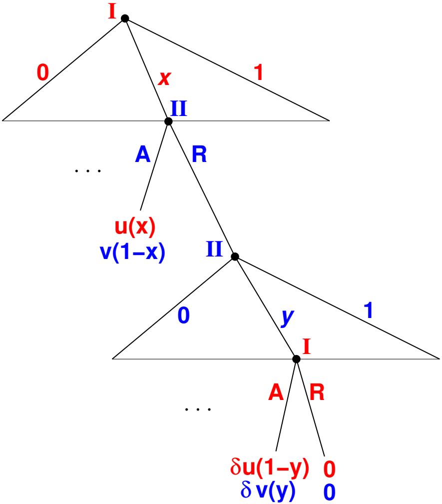
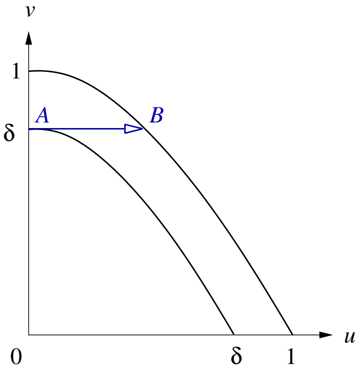


Graphical solution for two rounds

SPNE : in last round, player II makes the ultimatum demand of $y = 1$, player I accepts, player II gets $\delta v(y) = \delta$, player I gets 0.

In previous (first) round, player I makes player II indifferent between accepting and (A) rejecting and making her counterdemand, where she gets δ, by offering $1 - x$ so that (B) $\boxed{v(1-x) = \delta}$, and player II accepts in round 1, at point B.
Payoffs are $u(x)$, $v(1-x)$.



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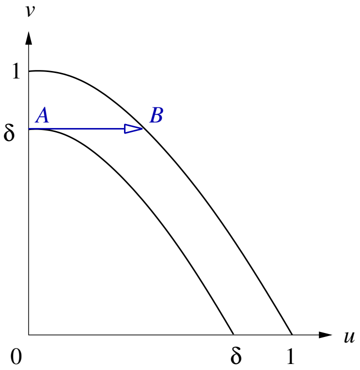
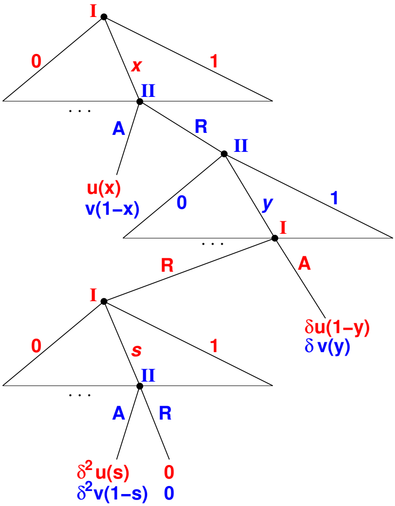


Bargaining in three rounds

x = demand by player I in round 1

y = counter-demand by player II in round 2

s = counter-counter-demand by player I in last round 3



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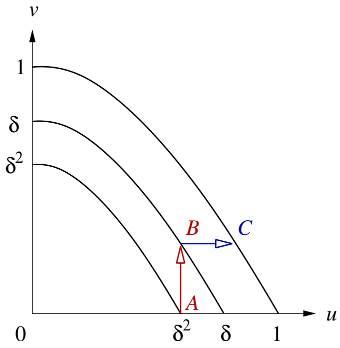

$A \to B: \delta^2 u(1) = \delta u(1-y)$ (round 2, player II chooses y)

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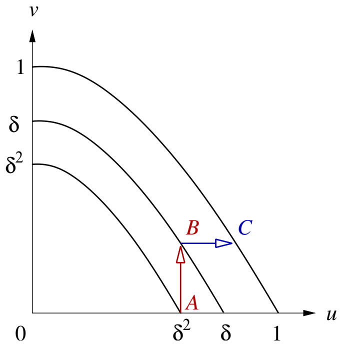

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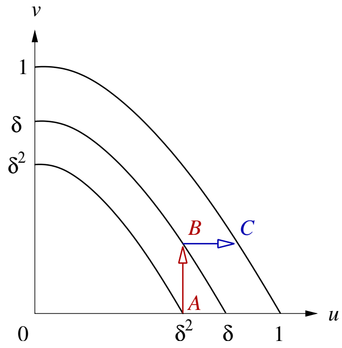

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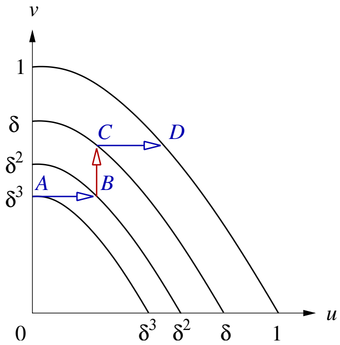

$A \to B:\ \delta^3 v(1) = \delta^2 v(1-s)$ (round 3, player I chooses s)

$B \to C:\ \delta^2 u(s) = \delta u(1-y)$ (round 2, player II chooses y)

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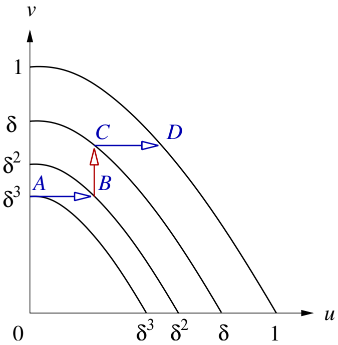
Graphical solution for four rounds



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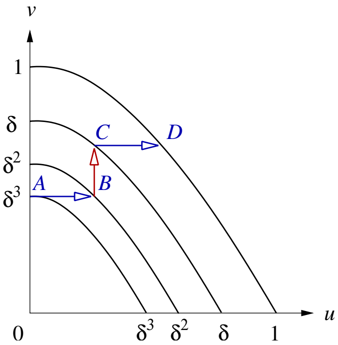

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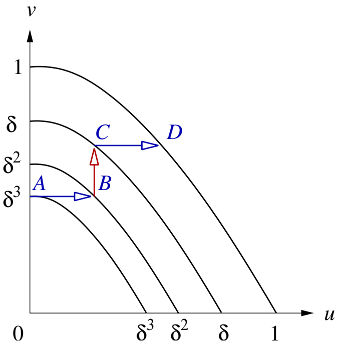

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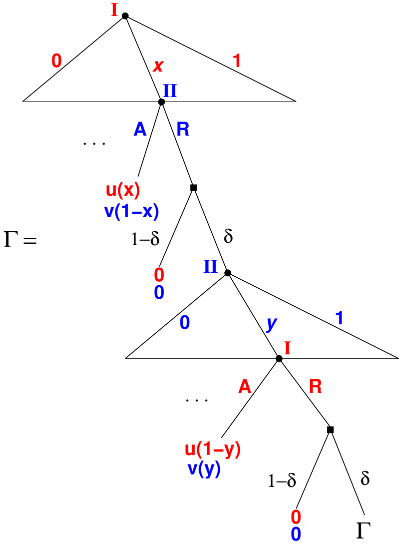
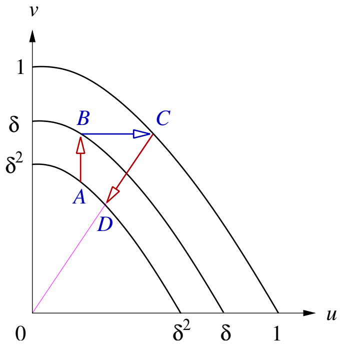
Infinite number of rounds

look for **stationary** strategies ***x*** and ***y***



Find stationary strategies graphically



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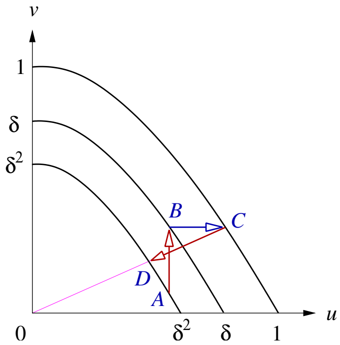

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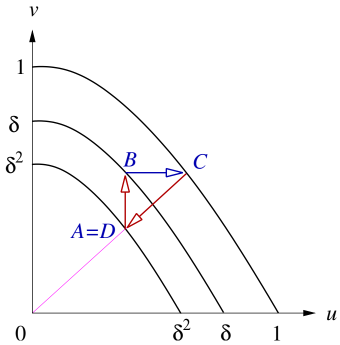

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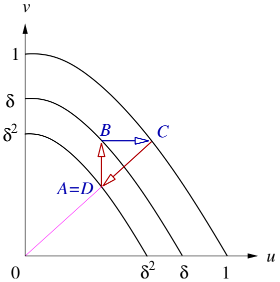
Characterization of stationary strategies



In rounds 2, 4, 6,...: $\boldsymbol{A} \rightarrow \boldsymbol{B}$: player II demands $\boldsymbol{y}$ so that $\boldsymbol{\delta^2 u(x) = \delta u(1-y)} \Leftrightarrow$
$\boxed{\boldsymbol{\delta u(x) = u(1-y)}}$

In rounds 1, 3, 5,...: $\boldsymbol{B} \rightarrow \boldsymbol{C}$: player I demands $\boldsymbol{x}$ so that $\boxed{\boldsymbol{\delta v(y) = v(1-x)}}$

(two equations with two unknowns)

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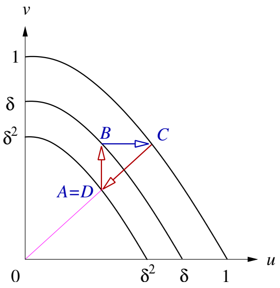


In rounds 2, 4, 6,...: $A \to B$: player II demands y so that $\delta^2 u(x) = \delta u(1-y) \Leftrightarrow$ $\boxed{\delta u(x) = u(1-y)}$

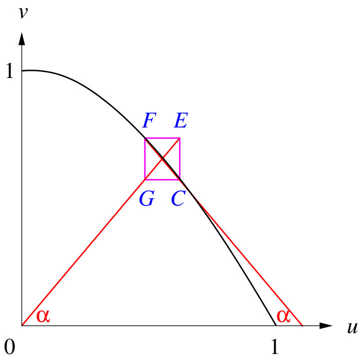
In rounds 1, 3, 5,...: $B \to C$: player I demands x so that $\boxed{\delta v(y) = v(1-x)}$
(two equations with two unknowns)

The Nash bargaining solution via alternating offers

Theorem

As $\delta \to 1$, the payoffs $u(x), v(y)$ for the stationary strategies x and y of alternating offers with an infinite number of rounds tend to the **Nash bargaining solution** U, V that maximizes UV for $U = u(x)$, $V = v(1 - x)$.

Graphical proof



$C = (u(x), v(1-x))$,
$F = (u(1-y), v(y))$,
$E = (u(x), v(y))$,
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$G \to C$:
$\boxed{\delta v(y) = v(1-x)}$,

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$\Rightarrow$ $CEFG$ is a **rectangle** with diagonals FC and GE of equal slope α.

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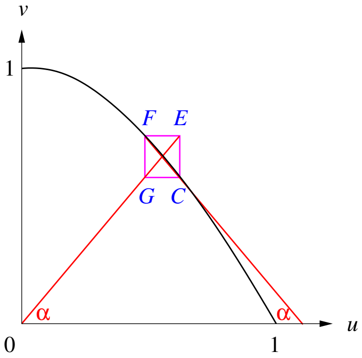

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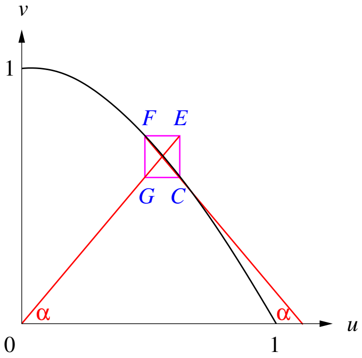

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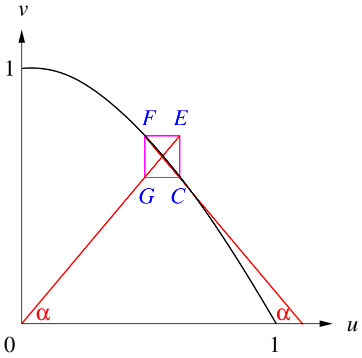

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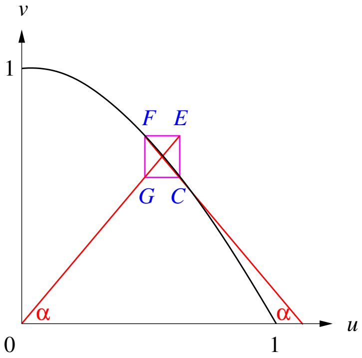

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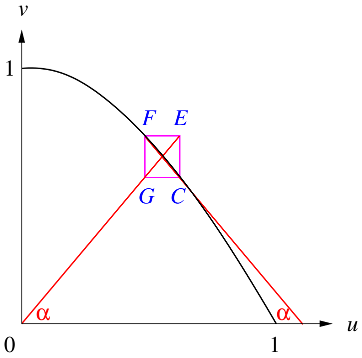

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Bargaining evidence from laboratory experiments

AE Roth (1995). 'Bargaining Experiments.' In Handbook of Experimental Economics, edited by John Kagel and Alvin E. Roth, 253-348. Princeton University Press.

VL Smith (1962). 'An Experimental Study of Competitive Market Behavior.' Journal of Political Economy 70(2): 111-137.

Ultimatum Game Bargaining

- recall last lecture

As in the Rubinstein bargaining model (with only one bargaining round)

1. the proposer (player 1) suggests a split between him and the receiver (player 2)
2. Player 2 can either accept or reject:
   1. If he accepts, the shares proposed by player 1 realize
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- **Nash equilibria:** any split supportable as a Nash equilibrium
- Unique **subgame-perfect Nash equilibrium** prediction: (1 all, 2 nothing)

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Recap: features and evidence

- Rejection by the responder kills own and other's payoff
- Any positive proposal, expecting acceptance, seems like a 'gift';
- however, expecting (off the SPNE-path) rejection if one's offer is too low, a substantial proposal may be strategically rational
- hence, for the responder, it may be rational to have a **rejection reputation**
- Meta-analysis suggests
  - proposals of roughly 40%;
  - high rejection rates for proposals under 20%, intermediate rejection rates for proposals of 20%-40%, and almost zero rejection rates for proposals >40%
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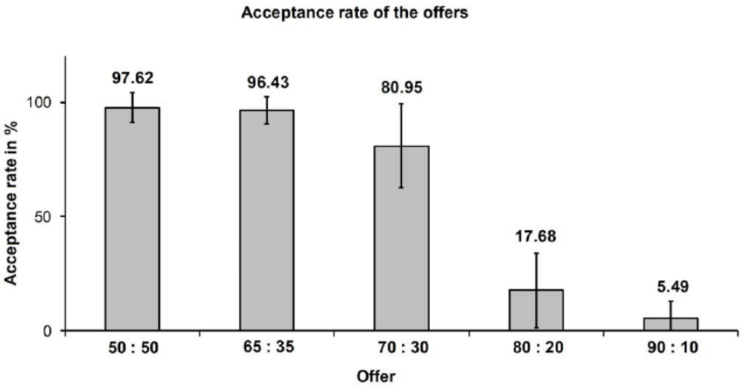
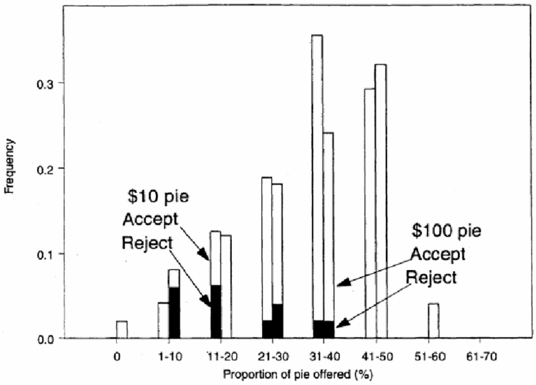
- Any positive proposal, expecting acceptance, seems like a 'gift';
- however, expecting (off the SPNE-path) rejection if one's offer is too low, a substantial proposal may be strategically rational
- hence, for the responder, it may be rational to have a **rejection reputation**
- Meta-analysis suggests
  - proposals of roughly 40%;
  - high rejection rates for proposals under 20%, intermediate rejection rates for proposals of 20%-40%, and almost zero rejection rates for proposals >40%
  - rates vary with stakes, matching protocol, etc.

Recap 1: acceptance rates



from Hollmann et al., PLoS ONE 2011

Recap 2: offers



from Hoffman et al., IJGT 1996

For one last time, let's play a large single-item economy

1. **Players:** All of you.
2. **Rules of the game:** See instructions.
3. Two individuals (who end up trading with each other) will be paid their payoffs in CHF.

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Thanks!

As always, please contact me under hnax@ethz.ch if you have questions.