NORMAL FORM GAMES: Equilibrium invariance and refinements

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Plan

- **Normal form games**
  - Equilibrium invariance
  - Equilibrium refinements

Nash's equilibrium existence theorem

Theorem (Nash 1951)

Every finite game has at least one [Nash] equilibrium in mixed strategies.

Cook book: How to find mixed Nash equilibria

- Find all pure strategy NE.

Check whether there is an equilibrium in which row mixes between several of her strategies:

- Identify candidates:
  - If there is such an equilibrium then each of these strategies must yield the same expected payoff given column's equilibrium strategy.
  - Write down these payoffs and solve for column's equilibrium mix.
  - Reverse: Look at the strategies that column is mixing on and solve for row's equilibrium mix.
- Check candidates:
  - The equilibrium mix we found must indeed involve the strategies for row we started with.
  - All probabilities we found must indeed be probabilities (between 0 and 1).
  - Neither player has a positive deviation.

Battle of the Sexes revisited

PLAYERS The players are the two students $N = \{row, column\}$.

STRATEGIES Row chooses from $S_{row} = \{Cafe, Pub\}$
Column chooses from $S_{column} = \{Cafe, Pub\}$.

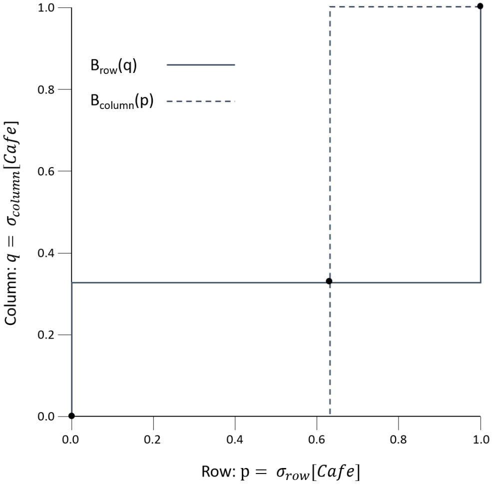
PAYOFFS For example, $u_{row}(Cafe, Cafe) = 4$. The following matrix summarises:

	$Cafe(q)$	$Pub(1-q)$	Expected
$Cafe(p)$	$\underline{4}, \underline{3}$	$1, 1$	$4q+(1-q)$
$Pub(1-p)$	$0, 0$	$\underline{3}, \underline{4}$	$3(1-q)$
Expected	$3p$	$p+4(1-p)$	

Column chooses $q = 1$ whenever $3p > p + 4(1-p) \Leftrightarrow 6p > 4 \Leftrightarrow p > \frac{2}{3}$.

Row chooses $p = 1$ whenever $4q + (1-q) > 3(1-q) \Leftrightarrow 6q > 2 \Leftrightarrow q > \frac{1}{3}$.

Battle of the Sexes: Best-reply graph



There is a mixed Nash equilibrium with $p = \frac{2}{3}$ and $q = \frac{1}{3}$.

Battle of the Sexes: Expected payoff

	Cafe$(1/3)$	*Pub*$(2/3)$	Expected
Cafe$(2/3)$	$4, 3$	$1, 1$	$4 \cdot 1/3 + 2/3$
Pub$(1/3)$	$0, 0$	$3, 4$	$3 \cdot 2/3$
Expected	$3 \cdot 2/3$	$2/3 + 4 \cdot 1/3$	

Frequency of play:

	Cafe$(1/3)$	*Pub*$(2/3)$
Cafe$(2/3)$	$2/9$	$4/9$
Pub$(1/3)$	$1/9$	$2/9$

Expected utility to row player: 2

Expected utility to column player: 2

Example

	L	R
T	$0, 0$	$\underline{3}, \underline{5}$
B	$\underline{2}, \underline{2}$	$\underline{3}, 0$

There are two pure-strategy Nash equilibria, at (B, L) and (T, R).

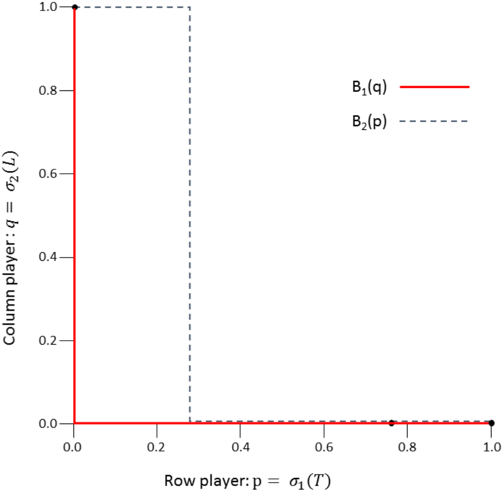
If row player places probability p on T and probability $1 - p$ on B.

$\Rightarrow$ Column player's best reply is to play L if $2(1 - p) \geq 5p$, i.e., $p \leq \frac{2}{7}$.

If column player places probability q on L and $(1 - q)$ on R.

$\Rightarrow$ B is a best reply. T is only a best reply to $q = 0$.

The best-reply graph



There is a *continuum* of mixed equilibria at $\frac{2}{7} \leq p \leq 1$, all with $q = 0$.

Example: Expected payoffs of mixed NEs

	L	R
T	$0, 0$	$\underline{3}, \underline{5}$
B	$\underline{2}, \underline{2}$	$\underline{3}, 0$

Frequency of play:

	$Cafe(0)$	$Pub(1)$
$Cafe(p > 2/7)$	0	p
$Pub(1-p)$	0	$1-p$

Expected utility to row player: 3

Expected utility to column player: $5 \cdot p \in (10/7 \approx 1.4, 5]$

Weakly and strictly dominated strategies

	L	R
T	$0, 0$	$\underline{3}, \underline{5}$
B	$\underline{2}, \underline{2}$	$\underline{3}, 0$

Note that T is *weakly dominated* by B.

- A weakly dominated pure strategy may play a part in a mixed (or pure) Nash equilibrium.
- A strictly dominated pure strategy cannot play a part in a Nash equilibrium!
  - Any mixed strategy which places positive weight on a strictly dominated pure strategy is itself strictly dominated. This can be seen by moving weight away from the dominated strategy.

Odd number of Nash equilibria

Theorem (Wilson, 1970)

Generically, any finite normal form game has an odd number of Nash equilibria.

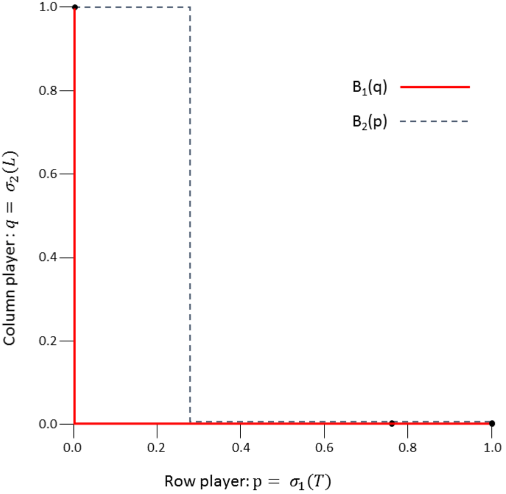
“Generically” = if you slightly change payoffs the set of Nash equilibria does not change.

Returning to our example

	L	R
T	$0, 0$	$\underline{3}, \underline{5}$
B	$\underline{2}, \underline{2}$	$\underline{3}, 0$

There are two pure-strategy Nash equilibria, at (B, L) and (T, R).
There is a *continuum* of mixed equilibria at $\frac{2}{7} \leq p \leq 1$, all with $q = 0$.

The best-reply graph



There is a *continuum* of mixed equilibria at $\frac{2}{7} \leq p \leq 1$, all with $q = 0$.

Example: Expected utility of mixed NEs

	L	R
T	$0, 0$	$\underline{3.1}, \underline{5}$
B	$\underline{2}, \underline{2}$	$\underline{3}, 0$

There are two pure-strategy Nash equilibria, at (B, L) and (T, R).

If row player places probability p on T and probability $1 - p$ on B.

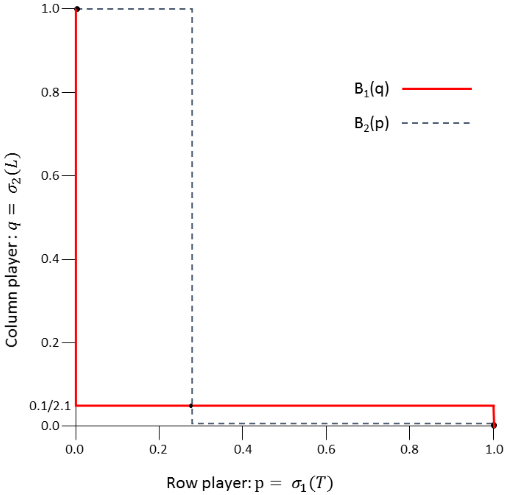
$\Rightarrow$ Column player's best reply is to play L if $2(1 - p) \geq 5p$, i.e., $p \leq \frac{2}{7}$.

If column player places probability q on L and $(1 - q)$ on R.

$\Rightarrow$ Row player's best reply is to play T if $3.1(1 - q) \geq 2q + 3(1 - q)$, i.e., $q \leq 0.1/2.1$.

The unique mixed strategy equilibrium is where $p = 2/7$ and $q = 0.1/2.1$.

The best-reply graph



There is a an odd number of equilibria.

Coordination game

	Email	*Fax*
Email	$\underline{5}, \underline{5}$	$1, 1$
Fax	$0, 0$	$\underline{3}, \underline{4}$

The two pure Nash equilibria are $\{Email, Email\}$ and $\{Fax, Fax\}$.

The unique mixed equilibrium is given by row player playing $\sigma_1 = (1/2, 1/2)$ and column player playing $\sigma_2 = (2/7, 5/7)$

Invariance of Nash equilibria

Proposition

Any two games G, G' which differ only by a positive affine transformation of each player's payoff function have the same set of Nash equilibria.

Adding a constant c to all payoffs of some player i which are associated with any fixed pure combination s_i for the other players sustains the set of Nash equilibria.

Coordination game

Now apply the transformation $u' = 2 + 3 \cdot u$ to the row player's payoffs:

	Email	*Fax*
Email	$\underline{5}, \underline{5}$	$1, 1$
Fax	$0, 0$	$\underline{3}, \underline{4}$

	Email	*Fax*
Email	$\underline{17}, \underline{5}$	$5, 1$
Fax	$2, 0$	$\underline{11}, \underline{4}$

The two pure Nash equilibria remain $\{Email, Email\}$ and $\{Fax, Fax\}$.

The unique mixed equilibrium is again given by row player playing $\sigma_1 = (1/2, 1/2)$ and column player playing $\sigma_2 = (2/7, 5/7)$

Some remarks on Nash equilibrium

Nash equilibrium is a very powerful concept since it exists (in finite settings)!

But there are often a multitude of equilibria. Therefore game theorists ask which equilibria are more or less likely to be observed.

We will focus next on a static refinements, strict and perfect equilibrium.

Later we will talk about dynamic refinements.

Strict Nash equilibria

Definition: Strict Nash Equilibrium

A *strict Nash equilibrium* is a profile σ^* such that,

$$U_i(\sigma_i^*, \sigma_{-i}^*) > U_i(\sigma_i, \sigma_{-i}^*) \text{ for all } \sigma_i \text{ and } i.$$

Perfect equilibrium or "trembling hand" perfection

Selten: 'Select these equilibria which are robust to small "trembles" in the player's strategy choices'

Definition: ε-perfection

Given any $\varepsilon \in (0,1)$, a strategy profile σ is ε-perfect if it is interior ($x_{ih} > 0$ for all $i \in N$ and $h \in S_i$) and such that:

$$h \notin \beta_i(x) \Rightarrow x_{ih} \leq \varepsilon$$

Definition: Perfect equilibrium

A strategy profile σ is perfect if it is the limit of some sequence of ε_t-perfect strategy profiles x^t with $\varepsilon_t \rightarrow 0$.

Perfect equilibrium or "trembling hand" perfection

Example:

	L	R
T	$1,1$	$1,0$
B	$1,0$	$0,0$

There are two pure Nash equilibira B,L and T,L. The mixed equilibrium is such that column player plays L and row player plays any interior mix.

Only T,L is perfect.

Note that T,L is not strict.

Perfect equilibrium or “trembling hand” perfection

Proposition (Selten 1975)

For every finite game there exists at least one perfect equilibrium. The set of perfect equilibria is a subset of the set of Nash equilibria.

Proposition

Every strict equilibrium is perfect.

THANKS EVERYBODY

Keep checking the website for new materials as we progress:
http://gametheory.online/project_show/9